SOLUTION OF LINEAR BOUNDARY INVERSE
PROBLEMS OF HEAT CONDUCTION

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New methods of solving linear one-dimensional boundary inverse problems of heat conduction are proposed; the methods are convenient for practical realization.

§1. The solution of linear boundary inverse problems of heat conduction (IPHC) by various methods has been considered in a number of works (for example, [1, 6-8, 11-16]). However, the direct methods of solution [7, 8, 11-13] are limited by the large time step of the approximation and regularized methods [14-16] usually demand a computer with a large memory. The present work pursues an intermediate path, attempting to increase the stability of the IPHC solution while retaining the simplicity of the direct methods of solution. Many linear one-dimensional boundary IHPC reduce to an integral Volterra equation of the first kind of convolution type

$$u(t) = \int_0^t K(t - \tau) z_0(\tau) d\tau,$$ (1)

where u(t) and K(t) are known functions and $z_0(t)$ is unknown. This improper problem is solved using an integral equation with a numerical parameter by means of which in [1] the temperature-field time lag was taken into account. The introduction of the numerical parameter α, chosen appropriately, increases the stability of the IPHC solution, as is confirmed by the numerical calculations below.

The procedure for reducing an integral Volterra equation of the first kind to an equation of the second kind is applied to Eq. (2), giving the result

$$u(t) = \int_0^t K(t - \tau + \alpha) z_\alpha(\tau) d\tau, \ \alpha \geqslant 0,$$ (2)

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The procedure for reducing an integral Volterra equation of the first kind to an equation of the second kind is applied to Eq. (2), giving the result

$$K(\alpha) z_\alpha(t) + \int_0^t \frac{\partial K(t - \tau + \alpha)}{\partial t} z_\alpha(\tau) d\tau = u'(t).$$ (3)

The problem of solving this equation for $K(\alpha) \neq 0$ is proper. Introducing the function $\tilde{z}_\alpha(t) = z_\alpha(t) K(\alpha)$ reduces Eq. (3) to an integral Volterra equation of the second kind in the usual form

$$\tilde{z}_\alpha(t) + \frac{1}{K(\alpha)} \int_0^t \frac{\partial K(t - \tau + \alpha)}{\partial t} \tilde{z}_\alpha(\tau) d\tau = u'(t).$$ (3a)

Usually in heat-conduction problems $K(\alpha) \neq 0$ when $\alpha > 0$, $u(0) = 0$, and the functions u(t) and K(t) have continuous derivatives. Then Eq. (3a) has a continuous solution $z_\alpha(t)$ when $\alpha > 0$. If u(t) has no continuous derivative, it is possible to use the method of treating the input data outlined in [3], which ensures a uniform approximation to the function u(t) itself and its derivative. The well-known methods [4] may be used for the numerical solution of Eq. (3): iteration, least squares, quadrature formulas, collocation, Laplace transforms, etc.

The numerical parameter α such that z_α is a good approximation to the accurate solution z_0 is chosen according to the discrepancy principle [5]

$$\sum_j \left[u(t_j) - \int_0^{t_j} K(t_j - \tau + \alpha) z_0(\tau) d\tau \right]^2 = \delta_0^2, \qquad (4)$$

where t_j is any uniform division of the points on the interval $[0, t_{max}]$ and δ_0 is the mean-square error of the input data. If δ_0 is unknown, the quasioptimum-parameter method [5] may be recommended for the choice of $z_\alpha(t)$. Some large number α^* is chosen and a sequence of numbers $\alpha_l = \alpha^* \beta^l$ $(0 < \beta < 1; l = 0, 1, 2, \ldots)$ is constructed. The value α_{qo} of α is found from the condition

$$\min_l \sum_j [z_{\alpha_l}(t_j) - z_{\alpha_{l-1}}(t_j)]^2 \qquad (5a)$$

or from the condition of a minimum of the maximum deviation

$$\min_l \max_j |z_{\alpha_l}(t_j) - z_{\alpha_{l-1}}(t_j)|. \qquad (5b)$$

§2. Direct solution of Eq. (2) is preferable, since the sufficient condition for its solution is that u(t) be constant. In engineering practice, the solution of Eq. (1) is often found by a method involving the approximation of z_0 by a discontinuous step function. In this case Eq. (1) reduces to the solution of a system of linear algebraic equations with a lower triangular matrix, the elements of which are diagonally equal [6]

$$\sum_{l=1}^{k} d^*_{kl} z_0^{[l]} = u(t_k), \quad k = 1, 2, \ldots, N, \qquad (6)$$

$$z_0^{[l]} = \frac{z_0(t_l) + z_0(t_{l-1})}{2}, \quad d^*_{kl} = \int_0^H K[(k - l + 1)H - \xi] d\xi. \qquad (6a)$$

Here $H = t_{max}/N$, $t_k = kH$, and N is the number of intervals. The algebraic system in Eq. (6) may be solved using the inverse matrix $B^* = \{b^*_{kl}\}$ of Eq. (6), which is also a lower triangular matrix with diagonally equal elements

$$b^*_{kk} = \frac{1}{d^*_{kk}}; \quad b^*_{kl} = \frac{-1}{d^*_{kk}} \sum_{i=l}^{k-1} d^*_{ki} b^*_{il}, \quad k > l. \qquad (7)$$

Formulas for the solution of various specific IPHC by this method are given in [6-8]. The main deficiency of this method of solving Eq. (1) is the large value of the critical step H (or ΔFo) at which the matrix d^*_{kl} becomes poorly specified [6]. A method of calculating linear IPHC better than that outlined above will now be suggested. The basis of the method is to solve Eq. (2) with the numerical parameter α, approximating the solution z_α by a discontinuous step function. Set

$$z_\alpha^{[l]} = \frac{z_\alpha(t_l) + z_\alpha(t_{l-1})}{2},$$

$$d_{kl} = \int_{t_{l-1}}^{t_l} K(t_k - \tau + \alpha) d\tau = \int_0^H K[(k - l + 1)H - \xi + \alpha] d\xi. \qquad (8)$$

Approximation of Eq. (2) using Eq. (8) leads to a system of linear equations differing only in the matrix elements from the system in Eq. (6). Therefore, Eq. (2) may be solved using Eq. (7) for the inverse matrix $B = \{b_{kl}\} = D^{-1}$, replacing the matrix elements d^*_k by d_{kl}. Accordingly, various specific IPHC may be calculated by this method using the formulas in [6-8], making appropriate changes in the equations; for example, in all the formulas for the matrix elements d^*_{in} given in Table 1 of [6], the replacement of $(n - p)$ by $(n - p + \alpha/\Delta Fo)$ is sufficient to give the matrix elements d_{in} for Eq. (2).

A series of calculations is made by the proposed method, in accordance with the quasioptimal-parameter method. In most cases, α^* may be taken to be α_{max}, the value of α at which the kernel $K(\alpha)$ has a maximum. For each number α_l $(l = 0, 1, \ldots)$ the above method gives a solution $z_\alpha(t)$ of Eq. (2) and a quasioptimal value α_{qo} is chosen from Eq. (5a) or (5b). Thus, for the approximate solution of Eq. (1), Eq. (2) must be solved several times with different values α_l of the parameter α.
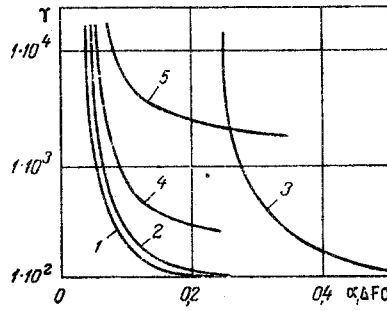
Fig. 1. Conditionality number γ for different methods of solving Eq. (9); 1, 2) the dependence $\gamma(\alpha)$ for the matrix of Eq. (8) with N = 20, $\Delta Fo_1 = 0.1$, and $\Delta Fo_2 = 0.05$; 3) the dependence $\gamma(\Delta Fo)$ for the matrix of Eq. (6) with N = 20; 4) the dependence $\gamma(\alpha)$ for the matrix of Eq. (8) with N = 20, Fo = 0.05; 5) the dependence $\gamma(\alpha)$ for the matrix of Eqs. (11) with $\Delta Fo = 0.1$, N = 10, m = r = 1, p = 0.

It may be expected that in the solution of Eq. (2) the permissible step H (or ΔFo) will be considerably reduced as a result of the regularizing effect of the numerical parameter α. To demonstrate this, consider the problem of heat-flux recovery at the boundary of a semiinfinite rod. In this case Eq. (2) takes the form

$$[u\,(\text{Fo}) - u\,(0)]\,\frac{\lambda}{x_1} = \frac{1}{\sqrt{\pi}}\int_0^{\text{Fo}} \exp\left[-1/4\,(\text{Fo} - \text{Fo}' + \alpha)\right]\frac{z_\alpha\,(\text{Fo}')\,d\text{Fo}'}{\sqrt{\text{Fo} - \text{Fo}' + \alpha}}\,. \tag{9}$$

The matrix elements d_{kl} are of the form

$$d_{kl} = 2\,\sqrt{\Delta Fo}\,(l - p + \alpha/\Delta Fo)i\Phi^*\left[\frac{1}{2\,\sqrt{\Delta Fo\,(l - p + \alpha/\Delta Fo)}}\right]_{p=k}^{p=k-1}\,. \tag{9a}$$

As in [6], the error in solving the algebraic system in Eq. (6) with the matrix D is estimated using the conditionality number $\gamma = \|D\| \cdot \|B\|$, which maximizes the spectral conditionality number of the matrix [6, 9, 10]. For N = 20, curve 3 in Fig. 1 shows the dependence of γ on the step ΔFo for Eq. (1), corresponding to the case $\alpha = 0$ in Eq. (9). Curves 1 and 2 of Fig. 1 show the dependence of γ on α for the steps $\Delta Fo_1 = 0.1$ and $\Delta Fo_2 = 0.05$, respectively, 5 and 10 times less than $\Delta Fo_{cr} = 0.5$ for $\alpha = 0$ [6]. Comparison of curves 1-3 shows in that Eq. (9) may be solved using the steps ΔFo_1 and ΔFo_2. Curve 4 in Fig. 1 shows the increase in the conditionality number γ with decrease in the approximation step by half, the total interval of variation of Fo being unchanged.

§3. The subject of this section is a method of solving Eq. (2) which is able to generate the function $z_\infty(t)$ with the first p continuous derivatives over the whole of the interval $t \in [0, t_{max}]$. The function $z_\alpha(t)$ is written in the form of a set of m-th order polynomials

$$z_\alpha\,(t) = \sum_{k=1}^{N} z_\alpha^{[k]}\,(t), \tag{10}$$

where

$$z_\alpha^{[k]}\,(t) = \begin{cases} \sum_{s=0}^{m} z_s^{[k]}\left(\dfrac{t - t_{k-1}}{H}\right)^s, & t \in [t_{k-1},\ t_k], \\ 0, & t \notin [t_{k-1},\ t_k]. \end{cases} \tag{10a}$$

In Eq. (10a) k gives the number of the interval $t \in [t_{k-1}, t_k]$. On the left-hand ends t_{k-1} (k = 2, ... , N) of the intervals $[t_{k-1}, t_k]$ the polynomials $z_\alpha^{[k]}(t)$ are joined either according to continuity (p = 0) or continuously with the first p (p = 1, 2, ... , m − p) derivatives.
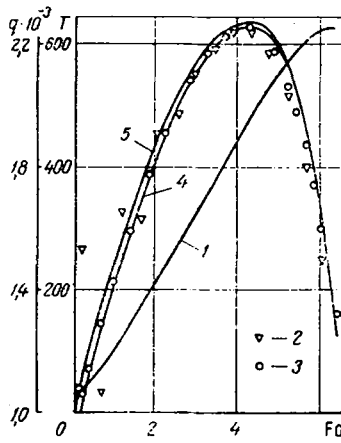
Fig. 2. Dependence of heat-flux recovery q(Fo), kW/m², for different methods of solution of the problem: 1) temperature T(Fo), °C, measured a distance $x_1 = 1.8 \cdot 10^{-3}$ m from the end of the rod; 2) points given by the formula of [8] for $\Delta Fo = 0.452$; 3) results of Eqs. (7) and (8) for $\Delta Fo = 0.1$; 4) continuous curve of q(Fo) plotted for $N = 2$, $r = 2$, $m = 3$, $p = 0$; 5) accurate solution of the problem.

To determine the coefficients $z_s^{[k]}$ the collocation method was applied to Eq. (2). On each interval $[t_{k-1}, t_k]$ the number of collocation points is $m - p$. The j-th collocation point $(j = 1, 2, \ldots, m - p)$ on the k-th interval will be denoted by $t_j^{[k]} = t_{k-1} + jh$; $h = H/(m - p)$. Taking into account the junction condition the functions $z_\alpha^{[k]}(t)$ and their derivatives up to and including the p-th order at the points t_{k-1} $(k = 2, 3, \ldots, N)$, a linear system of algebraic equations of $N(m + 1)$-th order is obtained for the calculation of the coefficients $z_s^{[k]}$. Its matrix has a cellular diagonal structure formed by cells of $(m + 1)$-th order

$$D_{11} Z^{[1]} = U^{[1]},$$
$$\sum_{l=1}^{k} D_{k-l} Z^{[l]} = U^{[k]}, \quad k = 2, 3, \ldots, N. \tag{11}$$

Here

$$Z^{[l]} = \begin{vmatrix} z_0^{[l]} \\ z_1^{[l]} \\ \cdots \\ \cdots \\ \cdots \\ z_{m-p}^{[l]} \end{vmatrix}, \quad U^{[k]} = \begin{vmatrix} 0 \\ \cdots \\ \cdots \\ 0 \\ u(t_1^{[k]}) \\ \cdots \\ \cdots \\ u(t_{m-p}^{[k]}) \end{vmatrix}, \quad 2 \leqslant k, \; l \leqslant N, \tag{11a}$$

$$D_0 = \begin{cases} \delta_{ns}(s-1)!, & n = 1, 2, \ldots, p+1, \\ b_{ns}^{[0]}, & n = p+1+j, \; j = 1, 2, \ldots, m-p, \end{cases}$$

$$D_1 = \begin{cases} (n-1)!/(s-1)!(n-s)!, & n = 1, 2, \ldots, p+1, \\ b_{ns}^{[1]}, & n = p+1+j, \; j = 1, 2, \ldots, m-p, \end{cases} \tag{11b}$$

$$D_{k-l} = \begin{cases} 0, & n = 1, 2, \ldots, p+1, \\ b_{ns}^{[k-l]}, & n = p+1+j, \; j = 1, 2, \ldots, m-p, \end{cases}$$
$$2 \leqslant k-l \leqslant N-1,$$

where δ_{ns} is the Kronecker delta, while

$$b_{ns}^{[0]} = \int_0^{jh} K(jh - \xi + \alpha)(\xi/H)^{s-1}d\xi,$$

$$b_{ns}^{[k-l]} = \int_0^{H} K[(k-l)H + jh - \xi + \alpha](\xi/H)^{s-1}d\xi,$$

$$1 \leqslant k - l \leqslant N - 1. \tag{11c}$$

On the first interval there is no junction condition. If r $(1 \leq r \leq m+1)$ collocation points $t_j^{[1]} = jh_1$ are chosen, where $h_1 = H/r$, then

$$\mathbf{Z}^{[1]} = \begin{vmatrix} z_0^{[1]} \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ z_{r-1}^{[1]} \\ 0 \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ 0 \end{vmatrix}, \quad \mathbf{U}^{[1]} = \begin{vmatrix} u(t_1^{[1]}) \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ u(t_r^{[1]}) \\ 0 \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ 0 \end{vmatrix},$$

$$D_{11} = \begin{cases} \tilde{b}_{ns}, & 1 \leqslant n \leqslant r, \\ 0, & r+1 \leqslant n \leqslant m+1, \end{cases} \tag{11d}$$

$$\tilde{b}_{ns} = \int_0^{nh_1} K(nh_1 - \xi + \alpha)(\xi/H)^{s-1}d\xi. \tag{11e}$$

For the practical calculation of the coefficients $z_s^{[k]}$ it is expedient to represent the system of $N(m+1)$-th order as a set of N algebraic systems of $(m+1)$-th order, differing only in the right-hand sides

$$D_{11}\mathbf{Z}^{[1]} = \mathbf{U}^{[1]}, \tag{12a}$$

$$D_0\mathbf{Z}^{[k]} = \mathbf{U}^{[k]} - \sum_{l=1}^{k-1} D_{k-l}\mathbf{Z}^{[l]}, \quad k = 2, 3, \ldots, N. \tag{12b}$$

The solution of Eq. (12) is given by the formulas

$$\mathbf{Z}^{[1]} = D_{11}^{-1}\mathbf{U}^{[1]}, \tag{13a}$$

$$\mathbf{Z}^{[k]} = D_0^{-1}\mathbf{U}^{[k]} - \sum_{l=1}^{k-1} D_0^{-1}D_{k-l}\mathbf{Z}^{[l]}, \quad k = 2, 3, \ldots, N. \tag{13b}$$

Numerical calculations by this method may be made on a computer with a small memory. Setting $p = 0$ and $m = 1$ reduces Eqs. (12b) to a particularly simple form.

It is of interest to analyze the dependence of the conditionality number γ on the parameter α in the problem of heat-flux recovery in a semiinfinite rod. Curve 5 in Fig. 1 corresponds to the presence of a single collocation point on each interval; in this case the solution $z_\alpha(t)$ of Eq. (9) is approximated by a constant, piecewise linear function. It is evident from a comparison of curves 1 and 5, constructed for the same order of the matrix and the same step ΔFo of the approximation, that the number of the system increases on passing from a discontinuous to a continuous approximating function.

Results given for the calculation of Eq. (9) by different methods for a specific dependence of the temperature T on the Fourier number ($\lambda = 1.085 \cdot 10^{-2}$ W/m·deg) are shown in Fig. 2. The numerical parameter α was chosen from Eq. (5a) for curve 3 and from Eq. (5b) for curve 4. These curves are in sufficiently good agreement with the accurate solution over the interval of variation of Fo, except in the region of very small Fo (Fo < α_{q0}), which indicates good accuracy of the heat-flux recovery. The practical realization of the proposed

methods of solution of linear boundary IPHC is fairly simple. In the considered example, finding a continuous solution (p = 0) of Eq.(9) using Eqs.(11)–(13) involves, in practice, the solution of two third-order algebraic systems.

NOTATION

λ	is the thermal conductivity;
x_1	is the coordinate of fixed thermal pickup;
ΔFo	is the Fourier-number increment;
$\| \cdot \|$	is the Euclidian norm of matrix.

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